It follows that V_n splits into the direct sum of the one-dimensional $\mathbf{T}$-invariant subspaces $\langle u_1^{n-k} u_2^k \rangle$, $k = 0, 1, \ldots, n$. Moreover, the corresponding subrepresentations of $\mathbf{T}$ are pairwise nonisomorphic. By Theorem 2 of 4.1, every $\mathbf{T}$-invariant subspace of V_n is a linear span of a number of monomials $u_1^{n-k} u_2^k$.

Now let W be an arbitrary nonnull SU_2-invariant subspace of V_n. By the foregoing discussion, W contains a monomial $u_1^{n-k} u_2^k$. Pick any nondiagonal matrix $A_0 \in SU_2$ and let it act on $u_1^{n-k} u_2^k$. It is readily seen that the coefficient of u_1^n in the form $f_0 = \Phi_n(A_0) u_1^{n-k} u_2^k$ is different from zero. Since $f_0 \in W$ and W is spanned by monomials, it follows that $u_1^n \in W$.

Analogously, considering the form $\Phi_n(A_0) u_1^n$, we remark that all its coefficients are different from zero. We thus conclude that all monomials belong to W, i.e., $W = V_n$, which completes the proof. □

Obviously

(8) $$\Phi_n(-E) = (-1)^n \varepsilon.$$

Hence, $-E$ belongs to the kernel of Φ_n if and only if n is even. For such values of n the representation Φ_n of SU_2 can be factored with respect to the normal subgroup $\{E, -E\}$, thereby yielding an irreducible representation of SO_3 that we will denote by Ψ_n.

Thus, for each integer $n \geq 0$ we have constructed an irreducible $(n+1)$-dimensional representation Φ_n of SU_2, and for each even $n \geq 0$, an irreducible $(n+1)$-dimensional representation Ψ_n of SO_3. In Section 11 we will show that *this is a complete list of the continuous irreducible complex representations of the groups* SU_2 *and* SO_3. (See also the Exercises in Section 8.)

Questions and Exercises

1. For arbitrary $A, B \in SU_2$ put

$$R(A, B)X = AXB^{-1} \qquad (X \in \mathbf{H}).$$

Show that R is a homomorphism of $SU_2 \times SU_2$ onto SO_4, and find its kernel.

2.* Let P be the linear representation of SU_2 constructed in 7.2. Construct an explicit isomorphism of the representations $P_{\mathbf{C}}$ and Φ_2 of SU_2.

3. Prove that any central function f on SU_2 is uniquely determined by its restriction to the subgroup

$$\mathbf{T} = \left\{ A(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbf{C},\ |z| = 1 \right\},$$

and that $f(A(z)) = f(A(z^{-1}))$.

4. Compute the restriction to $\mathbf{T}$ of the character χ_n of the representation Φ_n of SU_2.

5. Prove that the linear span of the functions

$$\phi_n(z) = \chi_n(A(z)) \qquad (z \in \mathbf{C},\ |z| = 1)$$

coincides with the space of all functions ϕ on the unit circle which can be written as polynomials in z and $\bar{z}$, and which satisfy the condition $\phi(\bar{z}) = \phi(z)$.

6. Let f be a continuous central function on SU_2. Show that

$$\int_{\mathrm{SU}_2} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(A(\mathrm{e}^{it}))\,\sin^2 t\,dt.$$

8. Matrix Elements of Compact Groups

In this section we generalize the main theorems established in Chapter II for finite groups to compact linear groups.

We shall consider only (continuous) *complex* linear representations. Recall that every complex representation of a compact group is unitary, and hence completely reducible (see Section 2).

8.1. Let X be a compact topological space on which integration is defined, i.e., there is given a positive linear functional

$$f \mapsto \int_X f(x)\,dx$$

on the space of continuous real-valued functions on X. We extend the integral by linearity to continuous complex-valued functions. Specifically, if $f = g+ih$, where g, h are continuous real-valued functions, we put

$$\int_X f(x)\,dx = \int_X g(x)\,dx + i\int_X h(x)\,dx.$$

Now, in the space of continuous complex functions on X we define a Hermitian inner product by the rule

$$(f_1, f_2) = \int_X f_1(x)\overline{f_2(x)}\,dx.$$

We let $C_2(X)$ denote the resulting (generally speaking, infinite-dimensional) Hermitian space.